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PRINCIPAL LINES ON SURFACES IMMERSED WITH CONSTANT MEAN CURVATURE

BY
C. GUTIÉRREZ AND J. SOTOMAYOR[1]

ABSTRACT. Configurations of lines of principal curvature on constant mean curvature immersed surfaces are studied. Analytical models for these configurations near general isolated umbilical points and particular types of ends are found. From the existence of transversal invariant measures for the foliations by principal lines, established here, follows that the union of recurrent lines of principal curvature is an open set. Examples illustrating all possible cases are given.

1. Introduction. This paper is devoted to the study of the possible configurations of lines of principal curvature and umbilical points of surfaces immersed in $\mathbf{R}^3$, with constant mean curvature. Except in the case where the image of the immersion is contained in a plane or a sphere, the umbilical points are isolated [**10, 11**]. A very precise analytical model is found for the family of principal lines near the isolated umbilical points. This model is related to a result of H. Hopf [**10, 11**], which permits the computation of the index of the umbilical point. The behaviour of the principal lines, at infinity near particular types of ends, called here elementary ends, of the immersion is also described through precise analytical models.

In connection with the global behaviour of the principal lines it is shown that their foliations have a natural smooth transversal invariant measure. This implies that the compact principal lines (cycles) appear in cylindrical open sets, and that the closures of recurrent principal lines have nonempty interior. Examples illustrating all possible situations are given. However the recurrent lines have been found only for immersions of tori into spaces which are not more than locally isometric to $\mathbf{R}^3$. The existence of such immersions into $\mathbf{R}^3$ seems to be a very difficult question related to a conjecture of H. Hopf [**24**, p. 684 (Problem 63)].[2]

From the point of view of principal lines, an immersion with constant mean curvature presents a global configuration which is quite opposite to that of the principally structurally stable immersions studied in [**4, 5**]. For a large class of this kind of immersions it is shown that the principal cycles are isolated and do not present recurrent lines [**4, 5**]. Examples of immersions of the sphere and torus into

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[2] H. Wente [**22**] has found counterexamples to this conjecture, all of which have closed principal lines.

$\mathbf{R}^3$, with nonconstant mean curvature and nontrivial recurrent principal lines have been given in [**5**].

The reader interested in the background for the study of configurations of lines curvature and umbilical points is referred to [**4, 10, 19**].

This paper is organized as follows. §2 contains the precise definitions of the objects involved in this work. §3 contains a review of the properties of isothermic coordinates that will be useful here. In §4 are found analytical normal forms for the principal lines near an umbilical point and an end of elementary type, for immersions with constant mean curvature. In §5 is shown the existence of natural tansversal smooth measures, invariant under the foliations $\mathscr{F}_\alpha$ and f_α defined by the principal lines of immersions with constant mean curvature. Important consequences of these measures on the structure of principal cycles and recurrent lines are also obtained here. In §6 are discussed three examples of immersions with constant mean curvature. In §7, the local results found in this work are used in the proof of a global decomposition theorem. Some problems related to the results obtained here are formulated in §8.

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The referee's suggestions were very helpful to us and have been incorporated into this work.

2. Preliminaries. Let M be an oriented connected two-dimensional smooth manifold and $\alpha: M \to \mathbf{R}^3$ be a smooth immersion. Call I_α and II_α respectively the first and second fundamental forms associated to α. $\mathrm{I}_\alpha = \alpha^*\langle\ ,\ \rangle$, where $\langle\ ,\ \rangle$ denotes the euclidean metric of $\mathbf{R}^3$. $\mathrm{II}_\alpha(a, b) = -\langle D\eta_\alpha(a), b\rangle$, where η_α is the positive Gaussian normal map of the immersion α. That is, η_α is the unitary vector in $\mathbf{R}^3$ in the direction of $\partial\alpha/\partial u \wedge \partial\alpha/\partial v$, where (u, v) is a positive chart of M and $\wedge$ denotes the exterior product associated to the positive orientation of $\mathbf{R}^3$. The maximal (resp. minimal) principal curvature of α at $p \in M$ is denoted $K_\alpha(p)$ (resp. $k_\alpha(p)$) and is defined to be the maximal (resp. minimal) *normal curvature* of α along all possible directions at p.

That is the maximum (resp. minimum) of $\mathrm{II}_\alpha(a, a)$ for $a \in T_pM$ and $\mathrm{I}_\alpha(a, a) = 1$.

The points $p \in M$ at which $K_\alpha(p) = k_\alpha(p)$ are called *umbilical points* of α. The set of these points will be denoted by $\mathscr{U}_\alpha$. Outside of $\mathscr{U}_c$, $K_\alpha > k_\alpha$ and the maximum $K_\alpha(p)$ (resp. minimum $k_\alpha(p)$) of the normal curvature is attained on the direction of a line $\mathscr{L}_\alpha(p)$ (resp. $l_\alpha(p)$) which is called the *maximal* (resp. *minimal*) *principal direction* of α at p. The integral lines of the smooth line field $\mathscr{L}_\alpha$ (resp. l_α) are called *maximal* (resp. *minimal*) *principal lines* of α, they fill up the open set $M - \mathscr{U}_\alpha$, defining the maximal (resp. minimal) principal foliation $\mathscr{F}_\alpha$ (resp. f_α) of α.

3. Isothermic coordinates. In local coordinates (u, v): $M \to \mathbf{R}^2$, $\mathrm{I}_\alpha = E_\alpha du^2 + 2F_\alpha du\,dv + G_\alpha dv^2$, and $\mathrm{II}_\alpha = L_\alpha du^2 + 2M_\alpha du\,dv + N_\alpha dv^2$.

The local coordinates (u, v) are called *isothermic* coordinates for α if $E_\alpha = G_\alpha$; $F_\alpha \equiv 0$. These coordinates always exist and are smooth [**19**]. The equation of principal lines of α in isothermic coordinates (u, v) is written

$$M_\alpha\, dv^2 + (L_\alpha - N_\alpha)\, du\, dv - M_\alpha\, du^2 = 0, \tag{1}$$

the *mean curvature*, defined by $\mathscr{H}_\alpha = \frac{1}{2}(K_\alpha + k_\alpha)$ is written

$$\mathscr{H}_\alpha(u, v) = (L_\alpha(u, v) + N_\alpha(u, v))/2E_\alpha(u, v), \tag{2}$$

and the equations of Codazzi become

$$\left(\frac{L_\alpha - N_\alpha}{2}\right)_u + (M_\alpha)_v = E_\alpha(\mathscr{H}_\alpha)_u, \quad \left(\frac{L_\alpha - N_\alpha}{2}\right)_v - (M_\alpha)_u = -E_\alpha(\mathscr{H}_\alpha)_v. \tag{3}$$

When $\mathscr{H}_\alpha$ is constant, these equations are precisely the Cauchy-Riemann equation for the complex function ϕ associated to the isothermic coordinates (u, v), which is defined by

$$\phi(u + iv) = ((L_\alpha - N_\alpha)/2)(u, v) - iM_\alpha(u, v), \tag{4}$$

and is therefore holomorphic.

It follows [**10**] that

$$|\phi|/E_\alpha = (K_\alpha - k_\alpha)/2. \tag{5}$$

Therefore, when $\mathscr{H}_\alpha$ is constant, the umbilical points are isolated unless the image of the immersion is contained in a plane or in a sphere [**10, 11**].

In terms of ϕ and $w = u + iv$, the equation (1) of principal lines is written

$$\operatorname{Im}\left[\phi(w)(dw)^2\right] = 0. \tag{6}$$

It follows that the index of an isolated umbilical point with complex coordinate $w = 0$, of an immersion with constant mean curvature, is equal to $-n/2$, where n is the order of the zero of ϕ at $w = 0$ [**10**].

If (u, v) and $(\tilde{u}, \tilde{v})$ are isothermic coordinates of α on a common domain of M, then the change of coordinates $\tilde{w} = \tilde{w}(w)$, where $\tilde{w} = \tilde{u} + i\tilde{v}$ and $w = u + iv$, is a holomorphic diffeomorphism. Conversely, if $\tilde{w} = \tilde{w}(w)$ is a holomorphic diffeomorphism and $w = u + iv$ define isothermic coordinates, for α, then $\tilde{w} = \tilde{u} + i\tilde{v}$ also does.

It follows [**10**] that the relation between the functions $\phi(u + iv)$ and $\tilde{\phi}(\tilde{u} + i\tilde{v})$ defined by (4) is

$$\phi(w) = \tilde{\phi}(\tilde{w}(w))\left(\frac{d\tilde{w}}{dw}(w)\right)^2. \tag{7}$$

4. Umbilical points and ends. Recall first, at least in partial form, a result originally due to Briot and Bouquet [**18**, p. 62; **21**, Chapter III], which will be useful in this section.

4.1 LEMMA. *Let $R(x, y)$ be a holomorphic function in a neighborhood of* $(0, 0) \in \mathbf{C}^2$ *with* $R(0, 0) = R_x(0, 0) = R_y(0, 0) = 0$.

If $b < 0$, the differential equation

$$x\frac{dy}{dx} = ax + by + R(x, y), \qquad y(0) = 0,$$

has a unique holomorphic solution $y = y(x)$ in a neighborhood of 0.

4.2 PROPOSITION. *Let p be an isolated umbilical point of an immersion α: $M \to \mathbf{R}^3$ with constant $\mathscr{H}_\alpha$. There are isothermic coordinates (u, v): $(M, p) \to (\mathbf{R}^2, 0)$ in which the associated complex function is given by $\phi(u + iv) = (u + iv)^n$, where $-n/2$ is the index of the umbilical point.*

PROOF. Let (x, y): $(M, p) \to (\mathbf{R}^2, 0)$ be isothermic coordinates for α and $\phi(x + iy)$ the associated complex function. Write $z = x + iy$ and $\phi(z) = cz^n(1 + r(z))$, where $c \neq 0$, n is a positive integer and r is holomorphic with $r(0) = 0$. By means of a linear transformation, the constant c can be taken equal to 1.

To find a local holomorphic diffeomorphism $\tilde{z} = \tilde{z}(z) = z(1 + \tilde{Z}(z))$, such that the complex function $\tilde{\phi}(\tilde{z})$ associated to the isothermic coordinates $\tilde{z}$ is given by $\tilde{\phi}(\tilde{z}) = (\tilde{z})^n$, it is equivalent, by (2.7), to solve the differential equation

$$\left(\frac{d\tilde{z}}{dz}(z)\right)^2 \tilde{\phi}(\tilde{z}(z)) = \phi(z). \tag{1}$$

In terms of $\tilde{Z}$ this equation becomes

$$\left(1 + \tilde{Z}(z) + z\frac{d\tilde{Z}(z)}{dz}\right)^2 z^n(1 + \tilde{Z}(z))^n = z^n(1 + r(z)), \tag{2}$$

which is equivalent to

$$z\frac{d\tilde{Z}(z)}{dz} = \frac{\sigma(1 + r(z)) - \sigma\left[(1 + \tilde{Z}(z))^{n+2}\right]}{\sigma\left[(1 + \tilde{Z}(z))^n\right]} \tag{3}$$

where σ is the holomorphic branch of the square root, with $\sigma(1) = 1$. By Lemma 4.1, this equation has a unique holomorphic solution $\tilde{Z}$ in a neighborhood of 0, with $\tilde{Z}(0) = 0$. Defining $u + iv = \tilde{z} = \tilde{z}(z) = z(1 + \tilde{Z}(z))$, the proof is finished.

4.3 COROLLARY. *Under the hypothesis of* 4.2, *there are $n + 2$ rays $L_0, L_1, \dots, L_{n+1}$ through $0 \in T_pM$, of which two consecutive rays make an angle of $2\pi/(n + 2)$. Tangent at p to each ray L_i there is exactly one maximal principal line S_i of $\mathscr{F}_\alpha$ which approaches p. Two consecutive lines S_i, S_{i+1}, $i = 0, 1, 2, \dots, n + 1$ $(S_{n+2} = S_0)$, bound a hyperbolic sector of $\mathscr{F}_\alpha$.*

The angular sectors bounded by L_i and L_{i+1} are bisected by rays l_i, $i = 0, 1, \dots, n + 1$, which play for f_α the same role as L_i for $\mathscr{F}_\alpha$. See Figure 1 *for an illustration. The lines S_i are called separatrices of $\mathscr{F}_\alpha$ at p. Similarly, for f_α.*

PROOF. It can be assumed that the equation of principal lines is given by $\operatorname{Im}[w^n(dw)^2] = 0$. By choosing a local orientation the tangent element to the principal lines, dw, at w can be assumed to satisfy $\arg(dw) = -n\theta/2$, for $\mathscr{L}_\alpha$, and $\arg(dw) = \pi/2 - n\theta/2$, for l_α, where $\theta \in [0, 2\pi]$ is the argument of w.

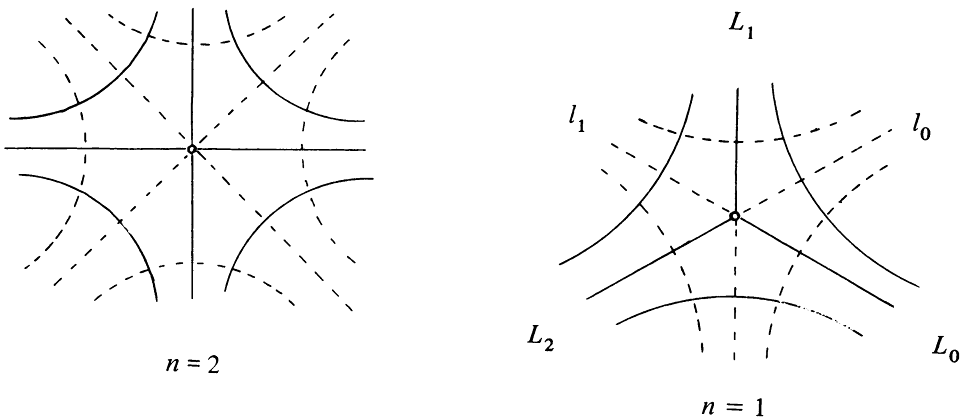

FIGURE 1

Consider the first case. The equation shows that arg(dw) is constant along the rays through 0. Therefore, the principal lines which are also rays are those whose arguments are solutions of $\theta = k\pi - n\theta/2$, $k \in N$ and $\theta \in [0, 2\pi]$, which are precisely $\theta_k = 2\pi k/(n+2)$, $k = 0, 1, \ldots, n+1, n+2$.

For $\theta \in (\theta_k, \theta_{k+1})$, the lines of $\mathscr{L}_\alpha$ are transversal to the ray of argument θ. The lines of $\mathscr{F}_\alpha$ therefore cross transversally all the rays contained in the angular sector bounded by θ_k, θ_{k+1}. From the fact that arg(dw) is decreasing with θ, follows that the rays L_k and L_{k+1} whose arguments are respectively θ_k and θ_{k+1}, bound a hyperbolic sector of the foliation $\mathscr{F}_\alpha$. These rays are separatrices which are the common boundary between consecutive hyperbolic sectors.

The case of the foliation f_α is similar.

4.4 LEMMA. *Let* $R(x, y)$ *be a holomorphic function in a neighborhood of* $(0,0) \in C^2$ *with* $R(0,0) = R_x(0,0) = R_y(0,0) = 0$. *If* $\alpha \in (1, \infty) - \mathbf{Z}$, *the differential equation*

$$x\frac{dy}{dx} = ax + \alpha y + R(x, y), \qquad y(0) = 0,$$

has a unique holomorphic solution $y = y(x)$ *in a neighborhood of* 0.

This lemma is in fact a corollary of the Poincaré-Dulac Theorem [**1**, Chapter 5] which implies that the vector field X

$$\dot{x} = x, \quad \dot{y} = ax + \alpha y + R(x, y)$$

is linearizable. The graph of the required holomorphic solution $y = y(x)$ is precisely the invariant manifold of X tangent to the x-axis at the origin.

4.5 PROPOSITION. *Let* (u, v): $M \to \mathbf{R}^2 - \{0\}$ *be isothermic coordinates for an immersion* α: $M \mapsto \mathbf{R}^3$ *with constant mean curvature. If the associated complex function* $\phi(w)$, $w = u + iv$, *has in* 0 *a pole of order* $n \in \{m \in \mathbf{N} | m$ *is either odd or* $m = 2\}$; *then there is a holomorphic diffeomorphism* $\tilde{w} = \tilde{w}(w)$, *with* $\tilde{w}(0) = 0$, *which defines isothermic coordinates* $(\tilde{u}, \tilde{v})$ *by* $\tilde{w} = \tilde{u} + i\tilde{v}$, *for which the associated complex function* $\phi(\tilde{u} + i\tilde{v})$ *reduces to* $a\tilde{w}^{-n}$, *where* $a = 1$ *if* $n \neq 2$ *and* $a = \lim_{z \to 0} z^2\phi(z)$ *if* $n = 2$.

PROOF. The function $\phi(w)$ can be written $bw^{-n}(1+R(w))$, with $R(0)=0$. To find holomorphic diffeomorphism $\tilde{w}=\lambda w(1+W(w))$, with $\lambda \neq 0$ and $W(0)=0$, which verifies the conclusion of the proposition, it is equivalent to solve the differential equation

$$(1)\quad \lambda^2\left(1+W(w)+w\frac{dW}{dw}(w)\right)^2 a\lambda^{-n}w^{-n}(1+W(w))^{-n}=bw^{-n}(1+R(w)).$$

Take λ so that $\lambda^{n-2}(b/a)=1$. This is possible because, when $n=2$, $b=a$. Therefore (1) is equivalent to

$$(2)\qquad w\frac{dW}{dw}(w)=\sigma[1+R(w)]\sigma\left[(1+W(w))^n\right]-(1+W(w)),$$

where σ is the holomorphic branch of the square root with $\sigma(1)=1$.

If $n=2$, $S(\xi)=[\sigma(1+R(\xi))-1]/\xi$ is holomorphic and equation (2) with $W(0)=0$ has the unique holomorphic solution

$$W(w)=\exp\left(\int_0^w S(\xi)\,d\xi\right)-1.$$

If $n\neq 2$, equation (2) can be written as

$$(3)\qquad w\frac{dW}{dw}=\alpha w+\frac{n-2}{2}W+R(w,W)$$

where $\alpha \in \mathbf{C}$ and $R(w,W)$ is a holomorphic function in a neighborhood of $(0,0)\in \mathbf{C}^2$ satisfying $R(0,0)=R_w(0,0)=R_W(0,0)=0$. If $n>2$ (resp. $n=1$) by Lemma 4.4 (resp. Lemma 4.1), there is a holomorphic diffeomorphism $W=W(w)$ defined in a neighborhood of $0\in\mathbf{C}$, with $W(0)=0$ and satisfying equation (3). This finishes the proof of the proposition.

As a consequence of the Poincaré-Dulac Theorem [**1**, Chapter 5], Proposition 4.5 cannot be extended to the case in which the order of the pole is even and greater than 2. However the following result can be stated.

4.6 PROPOSITION [**13**]. *Let* $(u,v)\colon M\to \mathbf{R}^2-\{0\}$ *be isothermic coordinates for an immersion* $\alpha\colon M\to\mathbf{R}^3$ *with constant mean curvature. If the associated complex function* $\phi(w)$, $w=u+iv$, *has in zero a pole of order* $n\neq 2$, *then there exists a small neighborhood* V *of* 0 *in* $\mathbf{R}^2$ *such that the principal lines of* α, *restricted to* $(u,v)^{-1}(V-\{0\})$, *distribute themselves* (*modulo topological equivalence*) *as if the associated complex function were* w^{-n}.

An *end* of the manifold M defined by the system of open sets

$$U_j=\{0<u^2+v^2<1/j,\ j\in N\}$$

where (u,v) are isothermic coordinates for α, on which the associated complex function has a pole of order n in $(u,v)=(0,0)$, is called an *elementary end* of α *of order* n. The index of such an elementary end is $n/2$. This elementary end is said to be *complete* if the distance to $(0,0)$ is infinite from any point in the punctured disk U_1. Examples of both complete and not complete, elementary ends are given in 6.2.

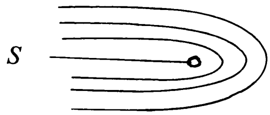

FIGURE 2.a

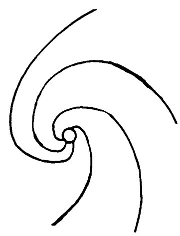

FIGURE 2.b.1

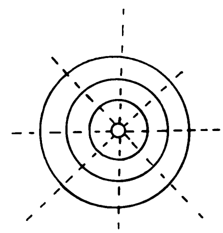

FIGURE 2.b.2

4.7 COROLLARY. *The lines of curvature of an immersion α with constant mean curvature near an elementary end E of order n are described as follows.*

(a) *For $n = 1$, there is exactly one line S (resp. s) of $\mathscr{F}_\alpha$ (resp. f_α) which tends to E, all the other lines fill a hyperbolic sector bounded by E and S (resp. s).*

(b) *For $n = 2$. Suppose that $\phi(z)$ is the associated complex and $a = \lim_{z \to 0} z^2\phi(z)$. There are two cases:*

(b.1) *$a \notin \mathbf{R} \cup (i\mathbf{R})$. Then the lines of $\mathscr{F}_\alpha$ and f_α tend to E.*

(b.2) *$a \in \mathbf{R} \cup (i\mathbf{R})$. Then the lines of $\mathscr{F}_\alpha$ (resp. f_α) are circles or rays tending to E and those of f_α (resp. $\mathscr{F}_\alpha$) are rays or circles.*

(c) *For $n \geqslant 3$, every line of $\mathscr{F}_\alpha$ and f_α tends to E. The principal lines distribute themselves into $n - 2$ elliptic sectors, two consecutive of which are separated by a parabolic sector.*

The proof of this corollary is similar to that of 4.3 for $n \neq 2$. In this case, $\arg(dw)$ is increasing with θ and the angular sectors bounded by θ_k, θ_{k+1} become elliptic instead of hyperbolic. For $n = 2$ the proof is immediate.

5. Transversal measures. The following lemma will be useful to prove a basic proposition of this section.

5.1 LEMMA. *Let $p \notin \mathscr{U}_\alpha$, where α is a smooth immersion of M into $\mathbf{R}^3$, with constant mean curvature. There are isothermic coordinates $(\tilde{u}, \tilde{v})$: $(M, p) \to (\mathbf{R}^2, 0)$ for which the associated complex function is $\tilde{\phi}(\tilde{u} + i\tilde{v}) \equiv 1$.*

PROOF. Take isothermic coordinates (u, v): $(M, p) \to (\mathbf{R}^2, 0)$. Since p is not umbilical the associated function $\phi(u + iv)$ satisfies $\phi(0) \neq 0$.

By means of a linear complex map it can be assumed that $\phi(0) = 1$.

Let $\tilde{w}(w) = \int_0^w \sigma \circ \phi(\xi)\, d\xi$ where σ is the holomorphic branch of the square root with $\sigma(1) = 1$. Clearly $w \to \tilde{w} = \tilde{u} + i\tilde{v}$ is a holomorphic diffeomorphism. By §3,

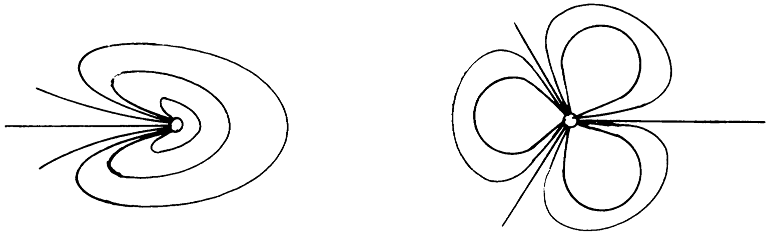

FIGURE 2.c

equation (7), the complex function $\tilde{\phi}(\tilde{u} + i\tilde{v})$ associated to the isothermic coordinates $(\tilde{u}, \tilde{v})$ satisfies

$$\phi(w) = \tilde{\phi}(\tilde{w})\left(\frac{d\tilde{w}}{dw}(w)\right)^2 = \tilde{\phi}(\tilde{w})\phi(w).$$

Therefore $\tilde{\phi}(\tilde{w}) \equiv 1$.

5.2 LEMMA. *Let Λ and λ be unitary tangent vector fields on an open set U of M. If $\alpha\colon M \to \mathbf{R}^3$ has constant mean curvature, and $\mathbf{R}\Lambda(p) = \mathscr{L}_\alpha(p)$, $\mathbf{R}\lambda(p) = l_\alpha(p)$ for every $p \in U - \mathscr{U}_\alpha$, then*

$$\Lambda/\left(\frac{K_\alpha - k_\alpha}{2}\right)^{1/2} \quad \textit{and} \quad \lambda/\left(\frac{K_\alpha - k_\alpha}{2}\right)^{1/2}$$

are commuting vector fields on U; that is, their Lie bracket is identically zero on U.

PROOF. Around any point $p \in U - \mathscr{U}_\alpha$ take isothermic coordinates (u, v) given by Lemma 5.1, i.e. with $\phi \equiv 1$. Then

$$\Lambda = \frac{\partial}{\partial u}/(E_\alpha)^{1/2}, \quad \lambda = \frac{\partial}{\partial v}/E_\alpha^{1/2}.$$

By §3, equation (5), $E_\alpha = 2/(K_\alpha - k_\alpha)$. Therefore

$$\frac{\partial}{\partial u} = \Lambda/\left(\frac{K_\alpha - k_\alpha}{2}\right)^{1/2} \quad \text{and} \quad \frac{\partial}{\partial v} = \lambda/\left(\frac{K_\alpha - k_\alpha}{2}\right)^{1/2}$$

are commuting.

The next proposition deals with transversal measures. A precise definition of this can be found in [7, Chapter I].

5.3 PROPOSITION. *Let $\alpha\colon M \to \mathbf{R}^3$ be an immersion with constant mean curvature $\mathscr{H}_\alpha$.*

Let F_1 and F_2 be lines of $\mathscr{F}_\alpha$ through p_1 and p_2 and let l be an oriented arc of the line of f_α with extremes p_1 and p_2. Let ds_1 and ds_2 be the elements of arc length on F_1 and F_2 oriented in such a way that the positive orientation of M at p_i is given by the product orientation of l and F_i.

The transformation π from F_1 to F_2 defined by the foliation f_α satisfies

$$(K_\alpha \circ \pi - \mathscr{H}_\alpha)^{1/2}\pi^* ds_2 = (K_\alpha - \mathscr{H}_\alpha)^{1/2} ds_1.$$

That is, π preserves the measure transversal to f_α defined by the form $(K_\alpha - \mathscr{H}_\alpha)^{1/2} ds$.

PROOF. Follows immediately from 5.2.

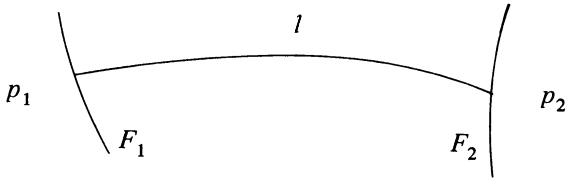

FIGURE 3

5.4 COROLLARY. *The principal cycles (closed principal lines) of an immersion with constant mean curvature appear in open sets.*

PROOF. Apply 5.3 to the case in which l is a cycle, $F = F_1 = F_2$ and $p_1 = p_2 = 0 \in F$. For $s_2 = \pi(s_1)$, it holds

$$\frac{ds_2}{ds_1} = \frac{(K_\alpha(s_1) - \mathscr{H}_\alpha)^{1/2}}{(K_\alpha(s_2) - \mathscr{H}_\alpha)^{1/2}}.$$

Therefore,

$$\int_0^{s_2} (K_\alpha(r_2) - \mathscr{H}_\alpha)^{1/2}\, dr_2 = \int_0^{s_1} (K_\alpha(r_1) - \mathscr{H}_\alpha)^{1/2}\, dr_1$$

and $s_2 = s_1 = \pi(s_1)$.

5.5 PROPOSITION. *Let α be a smooth immersion of M into $\mathbf{R}^3$, where $M = S - \{p_1, p_2, \ldots, p_n\}$, S is a compact connected oriented two-dimensional manifold and $\{p_i, i = 1, \ldots, n\}$ is a finite subset of S. Assume that α has finitely many umbilical points.*

If γ is a recurrent principal line of $\mathscr{F}_\alpha$ (resp. f_α) then the closure, $\bar{\gamma}$, of γ in $M - \mathscr{U}_\alpha$ is a two-dimensional submanifold of M whose boundary (when not empty) is made up of principal lines of $\mathscr{F}_\alpha$ (resp. f_α), with empty limit set (in $M - \mathscr{U}_\alpha$).

The proof of 5.5 is given below in 5.8 after some preliminary lemmas.

Let $\tilde{\mathscr{F}}_\alpha$ on $\tilde{M}$ be the orientable covering of $\mathscr{F}_\alpha$ on $M - \mathscr{U}_\alpha$; $\tilde{M}$ doubly covers $M - \mathscr{U}_\alpha$ and it has two components or one according as $\mathscr{F}_\alpha$ is orientable or not. Also $\tilde{M}$ is diffeomorphic to $\tilde{S} - \{q_1, q_2, \ldots, q_m\}$, where $\tilde{S}$ is a compact oriented two-manifold and $Q = \{q_1, q_2, \ldots, q_m\}$ is a finite subset of $\tilde{S}$. The points q_i will be the singularities of $\tilde{\mathscr{F}}_\alpha$ which is regarded, via a diffeomorphism, as a foliation in $\tilde{S} - Q$. The foliation $\tilde{\mathscr{F}}_\alpha$ will be endowed with a fixed orientation.

5.6. LEMMA [**3**, p. 312]. *Let $\tilde{\gamma}$ be a nontrivial recurrent line of $\tilde{\mathscr{F}}_\alpha$ and Σ be a cross section to $\tilde{\mathscr{F}}_\alpha$ passing through $\tilde{\gamma}$. If Σ is small enough, there exists a circle C transverse to $\tilde{\mathscr{F}}_\alpha$, passing through $\tilde{\gamma}$, such that any line which meets C also meets Σ. In particular, if $\tilde{\gamma}$ is dense in Σ, then it is also dense in C.*

5.7 LEMMA. *Let $C \subset \tilde{M}$ be a circle transverse to $\tilde{\mathscr{F}}_\alpha$. If T: $C \to C$ denotes the forward Poincaré return map induced by $\tilde{\mathscr{F}}_\alpha$, then, domain of T, when nonempty, is either the whole C or it consists of finitely many open intervals. Moreover if Σ is a* *connected component of the domain of T, then the frontier of $\Delta = \{\overrightarrow{pT(p)} / p \in \Sigma\}$ on $\tilde{S}$, consists of singularities of $\tilde{\mathscr{F}}_\alpha$ and arcs of lines of $\tilde{\mathscr{F}}_\alpha$ connecting them. Here $\overrightarrow{pT(p)}$ denotes the closed arc contained in the (oriented) line of $\tilde{\mathscr{F}}_\alpha$, which goes from p to $T(p)$.*

The proof of [**17**, Lemma 3] applies well to this lemma. The essential point is that $\tilde{\mathscr{F}}_\alpha$ has only finitely many singularities each of which has finitely many hyperbolic sectors [**6**, p. 161].

The configuration of $\tilde{\mathscr{F}}_\alpha$ restricted to some sets of the form: $\overline{\{\overrightarrow{pT(p)} | p \in \Sigma\}} - C$, where Σ is a connected component of the domain of T, is illustrated in Figure 4.

5.8 PROOF OF PROPOSITION 5.5. Consider the foliation $\tilde{\mathscr{F}}_\alpha$ on $\tilde{M}$. Denote by $\tilde{\gamma}$ one of the lines of $\tilde{\mathscr{F}}_\alpha$ which cover γ. The proof will follow immediately from:

(1) There exists a circle C transverse to $\tilde{\mathscr{F}}_\alpha$, passing through $\tilde{\gamma}$ such that $\overline{\tilde{\gamma} \cap C} = C$.

In fact, using (1), it may be observed, by Lemma 5.7, that the closure of the set made up of arcs of lines of $\tilde{\mathscr{F}}_\alpha$ which have their endpoints on C form a two-manifold whose boundary is made up of lines of $\tilde{\mathscr{F}}_\alpha$ which have empty limit set. Moreover the covering map of $\tilde{M}$ onto $M - \mathscr{U}_\alpha$ takes $\overline{(\tilde{\gamma})}$ onto $\bar{\gamma}$. Therefore, $\tilde{\gamma}$ will satisfy the required properties.

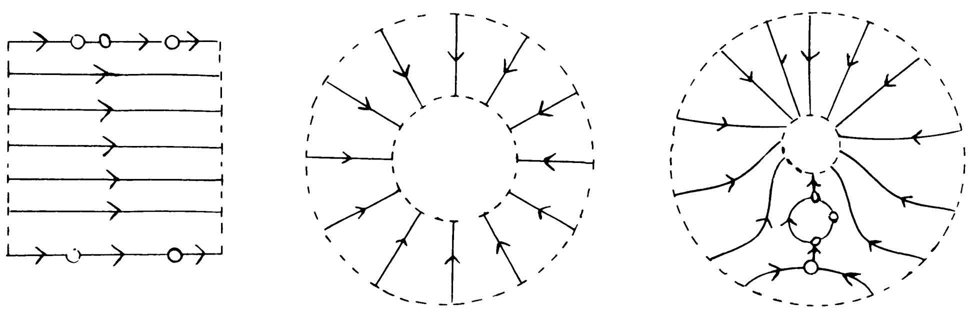

FIGURE 4

Now observe that (1) is equivalent to

(2) $\overline{C \cap \tilde{\gamma}}$ is not a Cantor set.

Certainly if $\overline{\tilde{\gamma} \cap C}$ has nonempty interior, considering a cross section contained on it and using Lemma 5.6, the circle C can be reconstructed so that the new circle, also denoted by C, verifies $\overline{\tilde{\gamma} \cap C} = C$. Therefore, using the fact that $\overline{\tilde{\gamma} \cap C}$ is a compact, perfect set, it results that $\overline{\gamma \cap C}$ is either a Cantor set or the whole C. This proves that (2) is equivalent to (1).

Now follows the proof of (2).

Suppose, by contradiction that $\overline{C \cap \tilde{\gamma}}$ is a Cantor set. A map $\tau \in \{T, T^{-1}\}$ induces in a natural way the map $\tilde{\tau}$: $\Omega \to \Omega$, where Ω is the set of connected components of $C - \overline{C \cap \tilde{\gamma}}$. It follows from 5.6 and 5.7 that

(3) there are at most finitely many elements $A_1, A_2, \ldots, A_n$ (respectively $B_1, B_2, \ldots, B_n$) of Ω which are not in the domain of $\tilde{T}$ (resp. $\widetilde{T^{-1}}$).

Since Ω is an infinite set, there must exist $H \in \Omega$ such that

(4) For some $\delta \in \{-1, 1\}$ and for all $j \in \mathbf{N}$, $\widetilde{T^{\delta j}}$ is defined on H.

Moreover, due to the fact that $\tilde{\mathscr{F}}_\alpha$ has a transversal invariant measure, induced by the transversal invariant measure of $\mathscr{F}_\alpha$ given in Proposition 5.3, and since the measure of each $\widetilde{T^{j\delta}}(H)$ must be positive, it results that

(5) There is some $m \in \mathbf{N}$ such that $\widetilde{T^{m\delta}}(H) = H$.

Notice that $\tilde{\gamma}$ accumulates on both endpoints of $T^{\delta j}(H)$, for all $j \in \mathbf{N}$. Thus, given $i \in \mathbf{N}$, by Lemma 5.7, there must exist two arcs made up of lines and singularities of $\tilde{\mathscr{F}}_\alpha$ which join the endpoints of $T^{\delta i}(H)$ and $T^{\delta(i+1)}(H)$. Therefore, using (5) there must exist a cycle, made up of arcs of lines and singularities of $\tilde{\mathscr{F}}_\alpha$, which is contained in the closure (in $\tilde{S}$) of $\tilde{\gamma}$. This is impossible by the Poincaré-Bendixson Theorem [**6**]. This proves (3) and ends the proof of 5.5.

6. Some examples.

6.1 *Example of isolated umbilical points for immersions with constant mean curvature*. The following version of the fundamental theorem of surface theory will be used in the construction of the example.

6.1.a THEOREM [**19**]. *Given analytic functions E, L, M, N, on an open simply connected set M of $\mathbf{R}^2$, there is an analytic immersion α: $M \to \mathbf{R}^3$ whose fundamental forms are*

$$\mathrm{I}_\alpha = E(du^2 + dv^2), \quad \mathrm{II}_\alpha = L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

for which (u, v) are isothermic coordinates, if and only if

(1) *$E > 0$ on M.*

(2) *The equations of Codazzi*

$$L_v - M_u = E_v\left(\frac{L+N}{2E}\right), \quad M_v - N_u = -E_u\left(\frac{L+N}{2E}\right)$$

are satisfied.

(3) *The equation of Gauss*

$$LN - M^2 = \frac{1}{2E}\left[E_u^2 + E_v^2 - EE_{uu} - EE_{vv}\right]$$

is satisfied.

The Gaussian and mean curvatures of α are given respectively by $\mathscr{K}_\alpha = (LN - M^2)/E^2$ and $\mathscr{H}_\alpha = (L + N)/2$.

6.1.b COROLLARY. *Let h be a real number. There is an immersion α with $\mathscr{H}_\alpha \equiv h$ and whose first and second fundamental forms are*

$$\mathrm{I}_\alpha = E(du^2 + dv^2), \quad \mathrm{II}_\alpha = L\,du^2 + 2M\,du\,dv + N\,dv^2$$

if and only if

(1) $E > 0$.

(2) $\phi(u + iv) = (L - N)/2 - iM$ *is holomorphic*, $L = \operatorname{Re}\phi + hE$ *and* $N = -\operatorname{Re}\phi + hE$.

(3) $E_u^2 + E_v^2 - EE_{uu} - EE_{vv} = -2E(|\phi|^2 - h^2E^2)$.

PROOF. Follows directly from 6.1.a.

The proof of the following proposition can also be found in [**8**, Corollary 5.2].

6.1.c PROPOSITION. *Given n a positive integer and h a real number, there are immersions α with* $\mathcal{H}_\alpha \equiv h$ *and isolated umbilical points of index* $-n/2$.

PROOF. Let $\phi(u+iv) = (u+iv)^n$. It will be proved that there is an analytic function $E = E(u,v)$, defined in a disk with center $0 \in \mathbf{R}^2$, such that

(1) $E > 0$,

(2) $E_u^2 + E_v^2 - EE_{uu} - EE_{vv} + 2E|\phi|^2 - 2h^2E^3 = 0$.

By the Cauchy-Kowalewsky Theorem [**19**, Volume 5] it follows that there are analytic functions U_1, U_2, U_3 which satisfy the following system of partial differential equations

$$\frac{\partial U_1}{\partial v} = U_2,$$
$$\frac{\partial U_2}{\partial v} = \frac{U_3^2}{U_1} + \frac{U_2^2}{U_1} - \frac{\partial U_3}{\partial u} + 2(u^2+v^2)^n - 2h^2U_1^2,$$
$$\frac{\partial U_3}{\partial v} = \frac{\partial U_2}{\partial u},$$

with the initial conditions $U_1(u,0) = U_2(u,0) \equiv 1$, $U_3(u,0) \equiv 0$.

Take $E = U_1$. It follows that $U_2 = E_v$. It will be shown that $U_3 = \partial U_1/\partial u = E_u$. In fact, using the partial differential equations, follows that

$$\begin{aligned} U_3(u,v) - \frac{\partial U_1}{\partial u}(u,v) &= U_3(u,v) - U_3(u,0) - \left(\frac{\partial}{\partial u}U_1(u,v) - \frac{\partial}{\partial u}U_1(u,0)\right) \\ &= \int_0^1 \frac{d}{dt}(U_3(u,tv))\,dt - \int_0^1 \frac{\partial}{\partial t}\left(\frac{\partial}{\partial u}U_1(u,tv)\right)dt \\ &= v\int_0^1 \left(\frac{\partial U_3}{\partial v}(u,tv) - \frac{\partial^2 U_1}{\partial v \partial u}(u,tv)\right)dt \\ &= v\int_0^1 \left(\frac{\partial U_2}{\partial u}(u,tv) - \frac{\partial U_2}{\partial u}(u,tv)\right)dt \equiv 0. \end{aligned}$$

Using Corollary 6.1.b follows that, defining

$$L = \operatorname{Re}\phi + hE, \quad N = -\operatorname{Re}\phi + hE \quad \text{and} \quad M = -\operatorname{Im}\phi,$$

there is an immersion α with constant mean curvature $\mathcal{H}_\alpha \equiv h$, with an umbilical point of index $-n/2$.

6.1.d REMARK. For $h = 0$ (minimal immersions) another proof of 6.1.c can be given. See 6.2.d below.

6.1.e REMARK. In [**23**], J. A. Wolf constructs examples of umbilic-free local surfaces of constant mean curvature.

6.1.f REMARK. The work of Kenmotsu [**15**] and of Hoffman-Osserman [**9**]—associating to a harmonic map into the two sphere a surface of constant mean curvature h with that map as its Gauss map—is relevant to the construction of examples here. In particular, if f is the Gauss map followed by stereographic projection then the Hopf function (i.e. the associated complex function) is given by $4fz\bar{f}z/h(1+|f|^2)^2$.

6.2 *Examples of immersions with elementary ends.* The following partial version of the Weierstrass representation theorem for minimal immersions will be useful in the discussion in the examples below.

6.2.a PROPOSITION [**16**, p. 64]. *Let D be an open connected set of $\mathbf{R}^2$. Let g be meromorphic and f be holomorphic on D. Assume that at each pole of order n of g the function f has a zero of order $2n$. Also that for each loop $\gamma \subset D$,* $\operatorname{Re}[\int_\gamma \phi_k(z)\,dz] = 0$, $k = 1, 2, 3$, *where*

$$\phi_1 = \frac{1}{2}f(1-g^2), \quad \phi_2 = \frac{i}{2}f(1+g^2), \quad \phi_3 = fg.$$

For a point $w_0 \in D$, define $\alpha_k(w) = \operatorname{Re}[\int_{w_0}^{w} \phi_k(z)\,dz]$, $k = 1, 2, 3$.

Then $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a minimal immersion (i.e. $\mathscr{H}_\alpha \equiv 0$) and (u, v), given by $w = u + iv$, are isothermic coordinates for α.

6.2.b COROLLARY. *Let α be given by* 6.2.a.

Then the complex function associated to the isothermic coordinates (u, v) is given by

$$\phi(w) = cf(w)g'(w)$$

where $c \in \{-1, 1\}$ is determined by the orientation of the immersed surface $\alpha(D)$.

PROOF. By definition of ϕ we have that

$$\begin{vmatrix} \phi_1' & \phi_2' & \phi_3' \\ \phi_1 & \phi_2 & \phi_3 \\ \operatorname{Im}\phi_1 & \operatorname{Im}\phi_2 & \operatorname{Im}\phi_3 \end{vmatrix} = c\phi E_\alpha$$

where $c \in \{-1, 1\}$ depends upon the chosen orientation of $\alpha(D)$. An easy computation shows us that

$$\begin{vmatrix} \phi_1' & \phi_2' \\ \phi_1 & \phi_2 \end{vmatrix} = ic\phi\phi_3.$$

It follows from this that $\phi = cfg'$. □

6.2.c EXAMPLE. For every positive integer n there are minimal immersions with a complete elementary end of index $n/2$. In fact:

The catenoid has an end of order 2, since one can write it using the Weierstrass representation with $f = 1$ and $g = 1/w$ on $\mathbf{C} - \{0\}$, which implies, by 6.2.b, that $\phi(w) = cw^{-2}$.

More generally, for $n \geqslant 2$, take $f(w) = w^{-n}$, $g(w) = w$ and $w_0 = 1$. By 6.2.b, $\phi(w) = cw^{-n}$ and by 4.4 the origin 0 is an elementary end of index $n/2$.

For n arbitrary take $f(w) = w^{-n}$, $g(w) = e^{\lambda w}$, where $\lambda \in \mathbf{C} - \{0\}$ is a constant. By 6.2.b $\phi(w) = c\lambda w^n e^{\lambda w}$. The index of 0 is determined by w^{-n}. As λ is arbitrary one gets examples of both cases of 4.7(b).

Using the fact that $E_\alpha = |f|^2(1 + |g|^2)^2/4$ (see [**16**]) it is easy to check that all of these ends are complete.

6.2.d EXAMPLE. For every positive integer $n \geqslant 1$ there are minimal immersions with not complete elementary ends of index $n/2$. Take $f(w) = we^{\lambda w^{-n}}/n$, $g(w) = e^{-\lambda w^{-n}}$ and $w_0 = 1$, where $\lambda \in \mathbf{C} - \{0\}$ is a constant. We may apply Proposition 6.2.a because

$$\int_{|w|=1} we^{\lambda w^{-n}}\,dw = i\int_0^{2\pi} e^{\lambda e^{-in\theta}}e^{2i\theta}\,d\theta$$

$$= -i\int_0^{-2\pi} e^{\lambda e^{in\theta}}e^{-2i\theta}\,d\theta = \int_{|w|=1} \frac{e^{\lambda w^{n}}}{w^3}\,dw = 0.$$

In this case $\phi(w) = \lambda w^{-n}$. These ends fail to be complete along the curves $\gamma(t) = (\lambda/i)^{1/n}t$, $t \in (0, \varepsilon)$. (Recall that $E_\alpha = |f|^2(1 + |g|^2)^2/4$.)

6.2.e EXAMPLE. For every positive integer there are minimal immersions with an umbilical point of index $-n/2$. In fact, take $f \equiv 1$, $g(w) = w^{n+1}/(n+1)$, $w_0 = 0$. Then, by 6.2.b $\phi(w) = cw^n$ and by (4.2) 0 is an umbilical point of α of index $-n/2$.

6.3 *Example of nontrivial recurrence*. Let G_θ be the action of the integers Z on $\mathbf{R}^3 = \mathbf{C} \times \mathbf{R}$, defined by $G_\theta(n, (z, s)) = (e^{2\pi n\theta i}z, s + n)$. Let π be the quotient mapping of $\mathbf{R}^3$ onto the quotient Riemannian manifold $\mathbf{R}^3_\theta = \mathbf{R}^3/G_\theta$. The foliation $\mathscr{F}_\alpha$ of the inclusion α of $T^2 = \pi(S^1 \times \mathbf{R})$ into $\mathbf{R}^3_\theta$, when θ is irrational, is an irrational foliation and their lines are dense in T^2.

7. A decomposition theorem.

7.1 THEOREM. *Let S be a compact connected oriented two dimensional manifold and $\{p_1, p_2, \ldots, p_n\} \subset S$. Let $\alpha\colon M \to \mathbf{R}^3$ be an immersion with constant mean curvature where $M = S - \{p_1, p_2, \ldots, p_n\}$. Then $M - \mathscr{U}_\alpha$ can be decomposed as the finite or countable union of the closure of open submanifolds M_i, pairwise disjoint, such that*:

(A) *Each $\partial M_i = \overline{M}_i - M_i$, when not empty, is made up of lines of $\mathscr{F}_\alpha$ which are separatrices of points of $\{p_1, p_2, \ldots, p_n\} \cup \mathscr{U}_\alpha$.*

(B) *If M_i contains a principal cycle of $\mathscr{F}_\alpha$, then it is filled up with principal cycles of $\mathscr{F}_\alpha$. Moreover, either $M = M_i = S$ is a torus or M_i is homeomorphic to a cylinder.*

(C) *If M_i contains a nontrivial recurrent line of $\mathscr{F}_\alpha$, then M_i is the interior of the closure (in $M - \mathscr{U}_\alpha$) of this line.*

(D) *If M_i contains no recurrence of $\mathscr{F}_\alpha$, then either $M = M_i = S - \{p_1, p_2\}$ and the foliation $\mathscr{F}_\alpha$ on M is topologically equivalent to the foliation on $\mathbf{R}^2 - \{(0,0)\}$ determined by the integral curves of $x(\partial/\partial x) + y(\partial/\partial y)$, or $\mathscr{F}_\alpha|M_i$ is topologically equivalent to the foliation on $\mathbf{R}^2$ determined by the integral curves of $\partial/\partial x$.*

PROOF. First, it will be proved that

(1) There are at most finitely many submanifolds M_i of M, satisfying (A) and such that $\mathscr{F}_\alpha|M_i$ is either as in case (B) or (C).

In fact, observe that the points of $\{p_1, p_2, \ldots, p_n\} \cup \mathscr{U}_\alpha$ appear as singularities of $\mathscr{F}_\alpha$. The local sector classification of the singularities of foliations [**6**, p. 161] says that each singularity of $\mathscr{F}_\alpha$ has only finitely many hyperbolic sectors. This implies (1) because each region M_i, when not equal to S, must intersect a hyperbolic sector of a singularity of $\mathscr{F}_\alpha$.

Let $\tilde{M}$ be the complement in M of the closure of all the regions M_i satisfying (1). Since these regions are finite, it follows that

(2) $\tilde{M}$ is homeomorphic to $\tilde{S} - \{q_1, \ldots, q_n\}$, where $\tilde{S}$ is a compact (not necessarily connected) oriented two dimensional manifold and $\{q_1, q_2, \ldots, q_n\} \subset \tilde{S}$.

It follows from 5.3, 5.4 and 5.5 that

(3) the limit set of any line of $\mathscr{F}_\alpha|\tilde{M}$ is empty. In fact, the lines of $\mathscr{F}_\alpha|\tilde{M}$ come from and go to singularities of $\mathscr{F}_\alpha$.

It remains to show that $\mathscr{F}_\alpha|\tilde{M}$ can be decomposed in regions satisfying (D). In fact, by (3) and [**6**, p. 161], the set Λ of separatrices of the singularities of $\mathscr{F}_\alpha|\tilde{M}$ is closed and any such separatrix is isolated from the others. Therefore, if M_i is a connected component of $M - \Lambda$, it can be constructed a global cross section to $\mathscr{F}_\alpha|M_i$ which meets each line at exactly one point. This section must be either a segment or a circle. Both of these cases lead to D. When $\mathscr{F}_\alpha$ is topologically equivalent to the foliation on $\mathbf{R}^2 - \{(0,0)\}$ determined by $x(\partial/\partial x) + y(\partial/\partial y)$, it follows from (3) and from the local configuration of $\mathscr{F}_\alpha$ at umbilics (Corollary 4.7), that all the lines of $\mathscr{F}_\alpha|M_i$ came from an end and go also to an end. The connectedness of S implies that $S = M_i$.

This finishes the proof.

7.2 REMARK. In Theorem 7.1, when the singularities $p_1, p_2, \ldots, p_n$ of $\mathscr{F}_\alpha$ have only finitely many separatrices, it follows from (A) that there are only finitely many submanifolds M_i. This happens for instance when each p_i is an elementary end of α. See 4.7.

8. Final remarks and related problems.

8.1 The existence of nontrivial recurrent principal lines for immersions into $\mathbf{R}^3$ with constant mean curvature, remains open in this paper. See Example 6.3, for immersions into manifolds locally isometric to $\mathbf{R}^3$.

8.2 It is shown in [**12**], examples of immersions of the three-sphere S^3 into the four-dimensional euclidean space E^4, which have constant mean curvature but are not round.

8.3 All of this work applies to the case of surfaces of constant mean curvature immersed in three-dimensional spaces of constant curvature. In fact, in this case, it has already been proved that the complex function associated to isothermic coordinates is holomorphic too; see the works of D. A. Hoffman [**8**, Lemma 2.1] and S. S. Chern [**2**, Theorem 1].

8.4 Are there elementary ends for immersion with constant nonzero mean curvature? See 6.2.c for minimal immersions. See also Remark 6.1.f.

More generally, the following considerably more difficult problem can be posed.

8.5 Let S be a compact, connected oriented, two-dimensional manifold and $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_l$ be points of S and $n_1, n_2, \ldots, n_k$; $m_1, m_2, \ldots, m_l$ be positive integers such that the Euler-Poincaré characteristic of S is equal to $-\frac{1}{2}\Sigma n_j + \frac{1}{2}\Sigma n_i$. Are there immersions α of $M = S - \{q_1, \ldots, q_l\}$ into $\mathbf{R}^3$, with constant mean curvature whose umbilical points are $p_1, \ldots, p_k$, with indexes $-n_1/2, \ldots, -n_k/2$ and whose ends are the points $q_1, \ldots, q_l$ which are elementary and have indexes $m_1/2, \ldots, m_l/2$?

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INSTITUTO DE MATEMÁTICA PURA E APLICADA, RIO DE JANEIRO, BRAZIL (Permanent address of both authors)

Current address (C. Gutierrez): Department of Mathematics 253-37, California Institute of Technology, Pasadena, California 91125